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The propagation of non-stationary waves from a spherical cavity in an acoustic layer☆

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ABSTRACT

The axisymmetric problem of the propagation of non-stationary waves from a spherical cavity in the plane of an infinite layer filled with an acoustic medium is considered. Using methods of incomplete separation of the variables in the space of Laplace transformation in time, the problem is reduced to an infinite system of linear algebraic equations, the solution of which is investigated in the form of series in exponential functions.

Similar problems were considered earlier in Refs 1 and 2 for non-classical regions.

1. Formulation of the problem

A spherical cavity of radius R is situated in a plane infinite acoustic layer of thickness $h+l$ ($h>R$, $l>R$). The centre O of the cavity is situated at distances of h and l respectively from the upper plane boundary $z=-h$ and the lower plane boundary $z=l$ of the layer. To investigate the motion of the medium we will use a spherical system of coordinates (r, θ, ϑ) with centre at the point O. The angle θ is measured from the positive direction of the Oz axis (Fig. 1).

The potential φ of the velocity υ in the acoustic medium, as is well known, satisfies the wave equation (dots denote differentiation with respect to the time τ)

$$\ddot{\varphi} = \Delta\varphi \tag{1.1}$$

An axisymmetric pressure $p_1(\tau, \theta)$ is applied to the surface of the cavity. We will write the corresponding boundary condition, taking into account the relation $p=-\dot{\varphi}$ between the pressure p in the medium and the potential, in the form

$$\dot{\varphi}|_{r=1} = -p_1(\tau, \theta) \tag{1.2}$$

It is assumed that the medium is at rest at the initial instant of time $\tau=0$, to which the following homogeneous initial conditions correspond

$$\varphi|_{\tau=0} = \dot{\varphi}|_{\tau=0} = 0 \tag{1.3}$$

and the solution of the problem is confined in the region $-h<z<l$, $r>1$.

The boundaries of the layer are assumed to be either free or absolutely stiff, so that, additionally taking into account the equality $\upsilon=\text{grad}\,\varphi$ and the initial conditions (1.3), we obtain the following boundary conditions:

$$\dot{\varphi}|_{z=-h} = 0 \ \text{ or } \ \partial\varphi/\partial z|_{z=-h} = 0 \tag{1.4}$$

$$\dot{\varphi}|_{z=l} = 0 \ \text{ or } \ \partial\varphi/\partial z|_{z=l} = 0 \tag{1.5}$$

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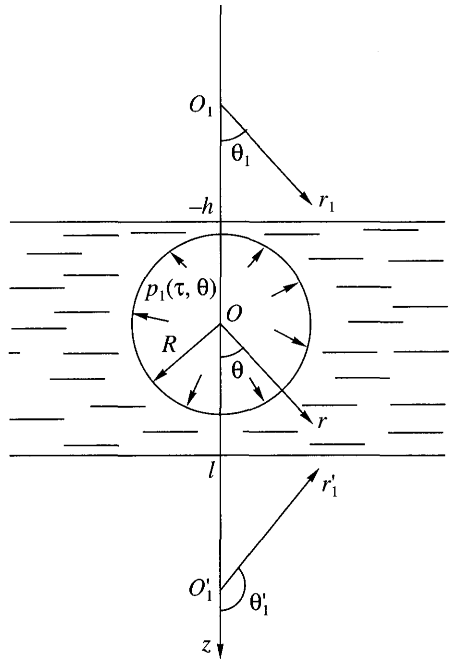

Fig. 1.

In the formulation of the problem (1.1)–(1.5) and henceforth we will use the following dimensionless quantities (dimensionless parameters are denote by a prime):

$$r = \frac{r'}{R},\quad z = \frac{z'}{R},\quad h = \frac{h'}{R},\quad l = \frac{l'}{R},\quad \tau = \frac{ct}{R},\quad p = \frac{p'}{\rho c^2},\quad \varphi = \frac{\varphi'}{cR}$$

where ρ and c are the density of the medium and the velocity of propagation of sound in it, and t is the dimensionless time.

2. Method of solution

We apply an integral Laplace transformation in time τ to initial-boundary-value problem (1.1)–(1.5) (the superscript L indicates a transformant and s is the transformation parameter). Then, the limited transform of the solution of Eq. (1.1), taking initial conditions (1.3) into account, can be represented in the form

$$\varphi^L(r, \theta, s) = r^{-1/2}\sum_{n=0}^{\infty} D_n(s)K_{n+1/2}(rs)P_n(\cos\theta) \tag{2.1}$$

where $K_{n+1/2}(x)$ are modified Bessel functions of the second kind, $P_n(x)$ are Legendre polynomials and $D_n(s)$ are unknown functions.

However, this form of the solution does not enable boundary conditions (1.4) and (1.5) to be satisfied on the plane boundaries. Hence, assuming $O = O_0 = O_0'$, $r = r_0 = r_0'$ and $\theta = \theta_0 = \theta_0'$, we additionally introduce two sequences ($i = 0, 1, 2, \ldots$) of spherical systems of coordinates $(r_i, \theta_i, \vartheta)$ and $(r_i', \theta_i', \vartheta)$ respectively with centres at the points O_i and O_i' such that the point O_{i+1}' is symmetrical to O_i relative to the plane $z = l$ and the point O_{i+1} is symmetrical to O_i' relative to the plane $z = -h$ (see Fig. 1). Hence it follows that the coordinates z_i and z_i' are the points O_i and O_i' satisfy the recurrence equations

$$z_{i+1} = -z_i' - 2h,\quad z_{i+1}' = -z_i + 2l,\quad z_0 = z_0' = 0 \tag{2.2}$$

Their solution has the form

$$z_i = \frac{1-(-1)^i}{2}(l-h) - i(l+h),\quad z_i' = \frac{1-(-1)^i}{2}(l-h) + i(l+h)$$

whence it follows that

$$z_{2k}' = -z_{2k} = 2k(l+h),\quad z_{2k+1} = z_{2k} - 2h,\quad z_{2k+1}' = z_{2k}' + 2l;\quad k = 0, 1, 2, \ldots \tag{2.3}$$

Then, the transform of the potential can be represented in the form of a series with terms of the form (2.1), corresponding to the indicated systems of coordinates:

$$\varphi^L = \sum_{i=0}^{\infty} r_i^{-1/2}\sum_{n=0}^{\infty} B_n^{(i)}(s)K_{n+1/2}(r_i s)P_n(\cos\theta_i) + \sum_{i=1}^{\infty} r_i'^{-1/2}\sum_{n=0}^{\infty} C_n^{(i)}(s)K_{n+1/2}(r_i' s)P_n(\cos\theta_i') \tag{2.4}$$

where $B_n^{(i)}(s)$, $C_n^{(i)}(s)$ are unknown functions. Hence we obtain the transform of the derivative of the potential

$$\frac{\partial\varphi^L}{\partial z} = \sum_{i=0}^{\infty} r_i^{-3/2}\sum_{n=0}^{\infty} B_n^{(i)}(s)L_n(r_is, \cos\theta_i) + \sum_{i=1}^{\infty} r_i'^{-3/2}\sum_{n=0}^{\infty} C_n^{(i)}(s)L_n(r_i's, \cos\theta_i')$$

$$L_n(x, y) = [nK_{n+1/2}(x) - xK_{n+3/2}(x)]yP_n(y) + K_{n+1/2}(x)(1-y^2)P_n'(y) \tag{2.5}$$

Now substituting Eqs (2.4) and (2.5) into boundary conditions (1.4) and (1.5), taking into account the relations

$$r_{i+1}\big|_{z=-h} = r_i'\big|_{z=-h}, \quad \theta_{i+1}\big|_{z=-h} + \theta_i'\big|_{z=-h} = \pi$$
$$r_{i+1}'\big|_{z=l} = r_i\big|_{z=l}, \quad \theta_{i+1}'\big|_{z=l} + \theta_i\big|_{z=l} = \pi; \quad i = 0, 1, 2, \dots \tag{2.6}$$

which follow from Eqs (2.3), and also the equality $P_n(-x) = (-1)^n P_n(x)$,[3] we obtain that these boundary conditions at are satisfied $z = -h$ and $z = l$, if we require that the arbitrary functions are relatedto one another as follows ($n \geq 0$):

$$B_n^{(i)} = \mp(-1)^n C_n^{(i-1)}, \quad i \geq 2, \quad B_n^{(1)} = \mp(-1)^n B_n^{(0)} \text{ when } z = -h \tag{2.7}$$

$$B_n^{(i)} = \mp(-1)^n C_n^{(i+1)}, \quad i \geq 0 \text{ when } z = l \tag{2.8}$$

The upper sign corresponds to the free boundary while the lower sign corresponds to the absolutely rigid boundary.

It follows from the system of recurrence equations (2.7) and (2.8) that all the arbitraryfunctions $B_n^{(i)}$ and $C_n^{(i)}$ are proportional to $B_n^{(0)}$:

$$B_n^{(i)} = C_n^{(i)} = \varepsilon_{in}^{(jk)} B_n^{(0)}, \quad i \geq 1, \quad n \geq 0$$

$$\varepsilon_{in}^{(11)} = \begin{cases} 1 \text{ when } i = 2q \\ (-1)^{n+1} \text{ when } i = 2q-1 \end{cases}, \quad \varepsilon_{in}^{(22)} = \begin{cases} 1 \text{ when } i = 2q \\ (-1)^{n} \text{ when } i = 2q-1 \end{cases}; \quad q = 1, 2, \dots$$

$$\varepsilon_{in}^{(12)} = (-1)^q \varepsilon_{in}^{(22)}, \quad \varepsilon_{in}^{(21)} = (-1)^q \varepsilon_{in}^{(11)} \tag{2.9}$$

The superscripts on the quantities $\varepsilon_{in}^{(jk)}$ have the following meaning: $j = 1$ ($j = 2$) corresponds to the first (second) condition of (1.4) and the upper (lower) sign in Eqs (2.7), while $k = 1$ ($k = 2$) corresponds to the first (second) condition (1.5) and the upper (lower) sign in Eqs (2.8).

Moreover, we will use the addition theorem[4] for $K_{n+1/2}(x)$ when $r < r_{0i}$

$$r_i^{-1/2}K_{n+1/2}(r_is)P_n(\cos\theta_i) = r^{-1/2}\sum_{m=0}^{\infty} T_{mn}(r_{0i}s)I_{m+1/2}(rs)P_m(\cos\theta), \quad i \geq 1$$

$$T_{mn}(s) = \sqrt{\frac{\pi}{2s}}(2m+1)\sum_{\sigma=|m-n|}^{m+n}(-1)^m b_\sigma^{(n0m0)} K_{\sigma+1/2}(s) \tag{2.10}$$

where r_{0i} is the distance between the points O and O_i, $b_\sigma^{(n0m0)}$ are the Clebsch-Gordon coefficients, and $I_{n+1/2}(x)$ is the modified Bessel function of the first kind.

Substituting expression (2.10) and its analogue for the system of coordinates (r_i', θ_i', ϑ) into Eq. (2.4) and taking relations (2.9) into account, we obtain the following expansion for the transform of the potential

$$\varphi^L = r^{-1/2}\sum_{n=0}^{\infty}\varphi_n^L(r, s)P_n(\cos\theta), \quad r < \min\{r_{01}, r_{02}, \dots; r_{01}', r_{02}', \dots\}$$

$$\varphi_n^L(r, s) = B_n^{(0)}(s)K_{n+1/2}(rs) + I_{n+1/2}(rs)\sum_{m=0}^{\infty} B_m^{(0)}(s)\sum_{i=1}^{\infty}\varepsilon_{im}^{(jk)}[T_{nm}(r_{0i}s) + T_{nm}(r_{0i}'s)] \tag{2.11}$$

where r_{0i}' is the distance between the points O and O_i'.

The quantities r_{0i}, r_{0i}' are obviously defined by the equations $r_{0i} = z_i$, $r_{0i}' = z_i'$ and, in accordance with Eqs (2.3), have the form

$$r_{00} = r_{00}' = 0, \quad r_{0,2k} = r_{0,2k}' = 2k(l+h), \quad r_{0,2k+1} = r_{0,2k} + 2h, \quad r_{0,2k+1}' = r_{0,2k} + 2l;$$
$$k = 0, 1, 2, \dots \tag{2.12}$$

Consequently, formula (2.11) holds when

$$r < \min\{r_{01}, r_{01}'\} = 2\min\{h, l\} \tag{2.13}$$

Taking into account the relations between the modified Bessel functions and elementary functions,[4] we represent the coefficients of series (2.11) as

$$\varphi_n^L(r,s) = \frac{1}{r^n s^{n+1}}\left[A_n(s)R_{n0}(rs)e^{-rs} + U_n(rs)\sum_{m=0}^{\infty} A_m(s)\sum_{i=1}^{\infty}\varepsilon_{im}^{(jk)}V_{nmi}(s)\right]$$
$$A_n(s) = \sqrt{\frac{\pi s}{2}}B_n^{(0)}(s), \quad U_n(s) = R_{n0}(-s)e^s - R_{n0}(s)e^{-s}$$
$$V_{nmi}(s) = S_{nm}(r_{0i}s)e^{-r_{0i}s} + S_{nm}(r'_{0i}s)e^{-r'_{0i}s}$$
$$R_{n0}(s) = \sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!2^k}s^{n-k}, \quad S_{nm}(s) = \frac{2n+1}{2s}\sum_{\sigma=|m-n|}^{m+n} b_\sigma^{(m0n0)}\frac{R_{\sigma0}(s)}{s^\sigma} \tag{2.14}$$

Now, expanding the transform of the pressure on the cavity surface in series in Legendre polynomials

$$p_1^L(s,\theta) = \sum_{n=0}^{\infty} p_{1n}^L(s)P_n(\cos\theta) \tag{2.15}$$

and taking relations (2.11) and (2.14) into account, we satisfy boundary condition (1.2). As a result, using formulae (2.12) we obtain an infinite system of linear algebraic equations in the unknown functions $A_n(s)$

$$\mathbf{F}(s)\mathbf{A}(s) = z^{-1}\mathbf{p}(s)$$
$$\mathbf{A}(s) = \|A_0(s), A_1(s), \ldots\|^{\mathrm{T}}, \quad \mathbf{p}(s) = \left\|p_{10}^L(s), \frac{s}{R_{10}(s)}p_{11}^L(s), \frac{s^2}{R_{20}(s)}p_{12}^L(s), \ldots\right\|^{\mathrm{T}}$$
$$\mathbf{F}(s) = \mathbf{E} + [z^{-2}\mathbf{M}(s) - \mathbf{E}]\times$$
$$\times\left\{2\sum_{q=1}^{\infty}\mathbf{Q}_{0q}(r_{0,2q}s)x^q y^q + \sum_{q=0}^{\infty}[\mathbf{Q}_{1q}(r_{0,2q+1}s)x^{q+1}y^q + \mathbf{Q}_{1q}(r'_{0,2q+1}s)x^q y^{q+1}]\right\}$$
$$\mathbf{Q}_{0q}(s) = \|\varepsilon_{2q,m}^{(jk)}, S_{nm}(s)\|, \quad \mathbf{Q}_{1q}(s) = \|\varepsilon_{2q+1,m}^{(jk)}, S_{nm}(s)\|$$
$$\mathbf{M}(s) = \left\|\frac{R_{n0}(-s)}{R_{n0}(s)}\delta_{nm}\right\|, \quad \mathbf{E} = \|\delta_{nm}\|, \quad x = e^{-2hs}, \quad y = e^{-2ls}, \quad z = e^{-s} \tag{2.16}$$

where $\mathbf{A}(s)$ and $\mathbf{p}(s)$ are infinite columns, $\mathbf{F}(s)$, $\mathbf{E}$, $\mathbf{M}(s)$, $\mathbf{Q}_{0q}(s)$ and $\mathbf{Q}_{1q}(s)$ are infinite matrices (the subscripts n and m correspond to the numbers of the rows and columns, $n, m = 0, 1, 2, \ldots$), and δ_{nm} is the Kronecker delta.

Analysing the scalar analogue of system (2.16), we expand its solution in the form of a series in exponential functions

$$\mathbf{A}(s) = \sum_{i,j,k=0}^{\infty}\mathbf{a}_{ijk}(s)x^i y^j z^{-2k-1}, \quad \mathbf{a}_{ijk}(s) = \|a_{ijk}^{(0)}(s), a_{ijk}^{(1)}(s), \ldots\|^{\mathrm{T}} \tag{2.17}$$

Substituting this series into system (2.16) and equating coefficients of like powers of the products of x, y and z we obtain that series (2.17) is the required solution if the columns $\mathbf{a}_{ijk}(s)$ satisfy the following recurrence relations

$$a_{000}^{(n)}(s) = -\frac{s^n}{R_{n0}(s)}p_{1n}^L(s), \quad \mathbf{a}_{00k}(s) = 0, \quad k \ge 1 \tag{2.18}$$

$$\mathbf{a}_{i00}(s) - \mathbf{Q}_{10}(r_{01}s)\mathbf{a}_{i-1,00}(s) = 0, \quad i \ge 1$$
$$\mathbf{a}_{i0k}(s) - \mathbf{Q}_{10}(r_{01}s)\mathbf{a}_{i-1,0k}(s) + \mathbf{M}(s)\mathbf{Q}_{10}(r_{01}s)\mathbf{a}_{i-1,0,k-1}(s) = 0, \quad i \ge 1, \quad k \ge 1 \tag{2.19}$$

$$\mathbf{a}_{0j0}(s) - \mathbf{Q}_{10}(r'_{01}s)\mathbf{a}_{0,j-1,0}(s) = 0, \quad j \ge 1$$
$$\mathbf{a}_{0jk}(s) - \mathbf{Q}_{10}(r'_{01}s)\mathbf{a}_{0,j-1,k}(s) + \mathbf{M}(s)\mathbf{Q}_{10}(r'_{01}s)\mathbf{a}_{0,j-1,k-1}(s) = 0, \quad j \ge 1, \quad k \ge 1 \tag{2.20}$$

$$\mathbf{a}_{ij0}(s) - 2\sum_{m=0}^{N_1(i,j)} \mathbf{Q}_{0,m+1}(r_{0,2m+2}s)\mathbf{a}_{i-m-1,j-m-1,0}(s) -$$

$$-\sum_{m=0}^{N_2(i,j)} \mathbf{Q}_{1m}(r_{0,2m+1}s)\mathbf{a}_{i-m-1,j-m,0}(s) - \sum_{m=0}^{N_3(i,j)} \mathbf{Q}_{1m}(r'_{0,2m+1}s)\mathbf{a}_{i-m,j-m-1,0}(s) = 0,\ i,j \geq 1$$

$$\mathbf{a}_{ijk}(s) + 2\sum_{m=0}^{N_1(i,j)} [\mathbf{M}(s)\mathbf{Q}_{0,m+1}(r_{0,2m+2}s)\mathbf{a}_{i-m-1,j-m-1,k-1}(s) -$$

$$-\mathbf{Q}_{0,m+1}(r_{0,2m+2}s)\mathbf{a}_{i-m-1,j-m-1,k}(s)] + \sum_{m=0}^{N_2(i,j)} [\mathbf{M}(s)\mathbf{Q}_{1m}(r_{0,2m+1}s)\mathbf{a}_{i-m-1,j-m,k-1}(s) - \tag{2.21}$$

$$-\mathbf{Q}_{1m}(r_{0,2m+1}s)\mathbf{a}_{i-m-1,j-m-1,k}(s)] + \sum_{m=0}^{N_3(i,j)} [\mathbf{M}(s)\mathbf{Q}_{1m}(r'_{0,2m+1}s)\mathbf{a}_{i-m-1,j-m,k-1}(s) -$$

$$-\mathbf{Q}_{1m}(r'_{0,2m+1}s)\mathbf{a}_{i-m-1,j-m-1,k}(s)] = 0;\quad i,j,k \geq 1$$

where

$$N_1(i,j) = \min(i-1,j-1),\quad N_2(i,j) = \min(i-1,j),\quad N_3(i,j) = \min(i,j-1) \tag{2.22}$$

Using induction over the sum of the indices $i+j$ with a basis in the form of relation (2.18) it can be shown that

$$\mathbf{a}_{ijk}(s) = 0 \text{ when } i+j>k \tag{2.23}$$

Hence it follows that $ih+jl-k>i+j-k>0$, and a certain right half-plane $\operatorname{Re} s>\alpha$ exists in which

$$\left|x^i y^j z^{-2k}\right| = \left|e^{-2(ih+jl-k)s}\right| < 1 \tag{2.24}$$

and series (2.17) converges in this half-plane.

Recurrence relations (2.18)–(2.21) enable us to determine all the elements of the corresponding columns $\mathbf{a}_{mkl}(s)$ without using reduction of the infinite system of equations. An analysis of these relations shows that the elements of the required columns are rational functions of the Laplace transformation parameter, which enables us to calculate their originals, and, consequently, also the originals of the coefficients of the series for the potential, the pressure p and the velocity of the medium using the theory of residues.

The expanded form of writing the transformation of the coefficients of the series in Legendre polynomials for the potential, in accordance with relations (2.14), (2.17) and (2.23), has the form

$$\varphi_n^L(r,s) = \frac{1}{r^n s^{n+1}} \sum_{\substack{l_1,l_2,l_3=0\\ l_1+l_2>l_3}}^{\infty} \left[a_{l_1l_2l_3}^{(n)}(s)R_{n0}(rs)e^{-rs} + \right.$$

$$\left. + U_n(rs)\sum_{m=0}^{\infty} a_{l_1l_2l_3}^{(m)}(s)\sum_{i=1}^{\infty} \varepsilon_{im}^{(jk)} V_{nmi}(s)\right] x^{l_1} y^{l_2} z^{-2l_3-1} \tag{2.25}$$

Here the transformation of the pressure, according to relations (2.11) and (2.14), is defined as

$$p^L = -sr^{-1/2}\sum_{n=0}^{\infty} \varphi_n^L(r,s)P_n(\cos\theta) \tag{2.26}$$

We will now show that a formula for the pressure in the analogous problem for a half-plane follows from the solution obtained. For this purpose we put $l=\infty$. Then, by relations (2.12), (2.14) and (2.16) in the above-mentioned right half-plane $y\to 0$, $r_{0i}'\to\infty$ when $i\geq 1$, $r_{0i}\to\infty$ when $i\geq 2$. In the limit we obtain that $\mathbf{Q}_{0q}(r_{0,2q}s)=\mathbf{Q}_{1q}(r_{0,2q+1}s)=0$ when $q\geq 1$ and $\mathbf{Q}_{1q}(r'_{0,2q+1}s)=0$ when $q\geq 0$. Then, it follows from relations (2.20) and (2.21) that $\mathbf{a}_{ijk}(s)=0$ when $j\geq 0$ and $i,k\geq 0$, and the non-trivial relations (2.20) and (2.21) define the coefficients

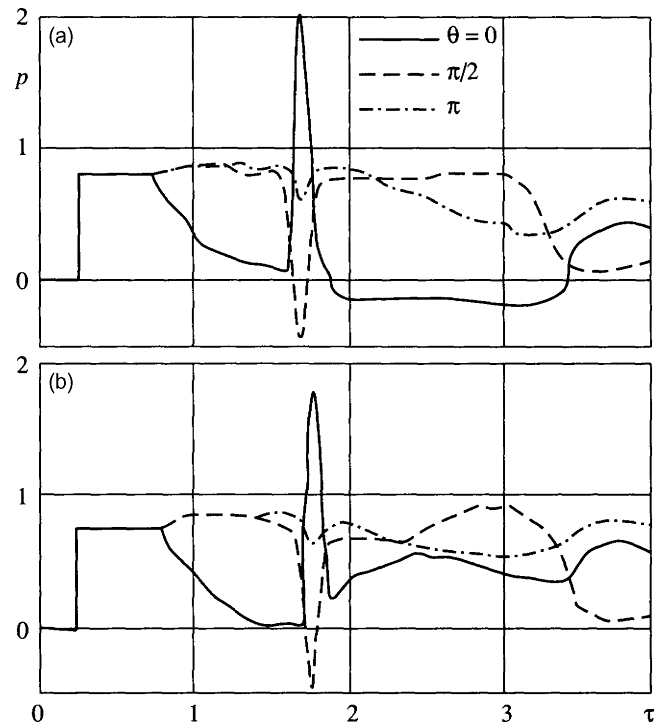

Fig. 2.

$a_{i0k}^{(n)}(s) = a_{ik}^{(n)}(s)$, which are independent of the index j. In this case formula (2.25) takes the form

$$\varphi_n^L(r, s) = \frac{1}{r^n s^{n+1}} \sum_{\substack{l_1, l_2 = 0 \\ l_1 > l_2}}^{\infty} \left[a_{l_1 l_2}^{(n)}(s) R_{n0}(rs) e^{-rs} + \right.$$

$$\left. + U_n(rs) \sum_{m=0}^{\infty} a_{l_1 l_2}^{(m)}(s) \varepsilon_{1m}^{(jk)} S_{nm}(r_{01} s) e^{-r_{01} s} \right] x^{l_1} z^{-2l_2 - 1} \tag{2.27}$$

which, apart from the notation, is identical with the solution of the problem of the propagation of non-stationary waves from a spherical cavity in an acoustic half-space.[1]

3. Example

We will consider the case when the pressure $p_1(\tau, \theta) = H(\tau)$ is uniformly distributed over the surface of the cavity, where $H(\tau)$ is the Heaviside function. We will take water ($c = 1500$ m/s, $\rho = 1000$ kg/m^3) as the acoustic medium. The geometry of the problem is $h = 1.5$ and $l = 2.0$.

In Fig. 2a we show graphs of the change in the pressure p with time at points of the medium corresponding to $r = 1.3$ and different values of θ for the following versions of the boundary conditions: $z = -h$ – the free surface and $z = l$ – the fixed surface. Similar relations for the case when both surfaces $z = -h$ and $z = l$ are free are shown in Fig. 2b.

We retained five terms of the series in Legendre polynomials for the calculations. Only a slight refinement of the results is obtained by taking into account a larger number of terms.

The discontinuities of the first kind on the graphs correspond to the first instant of the arrival of the spherical wave front, while the successive constant value of the pressure corresponds to the solution of the problem for unbounded space. The pressure spikes on the graphs for $\theta = 0$ and $\theta = \pi$ are probably due to the closeness of the corresponding points to the plane boundaries and the superposition of the waves reflected from these boundaries.

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